Lecture 8: Surfaces w/ $P_g = 0,\ q \geq 1$

Lemma 1: (a) $P_g = 0,\ q \geq 1 \Longrightarrow K_X^2 \leq 0$

$K_X^2 < 0$ unless $q = 1,\ b_2 = 2$

(b) X minimal surface w/ $K^2 < 0 \Longrightarrow P_g = 0,\ q \geq 1$.

pf: (a) $$\underset{1-q+\cancel{P_g}^{\,0}}{\chi(\mathcal{O}_X)} = \frac{1}{12}\big(K^2 + \underset{2-4q+b_2}{\chi_{top}(X)}\big)$$ Noether's formula

$$10 = K^2 + 8q + b_2 \quad \text{or} \quad K^2 = 10 - 8q - b_2$$

It suffices to show that $q=1,\ b_2=1$ won't happen.

Consider $X \xrightarrow{\alpha} C = \alpha(X)$ Albanese map

$q=1 \Rightarrow g_C = 1$

Let F be the class of α $\qquad F^2 = 0$

H = hyperplane section $\qquad H.F > 0$

Thus, $\mathbb{Z}\langle F, H\rangle \subseteq H^2(X,\mathbb{Z})$ $\quad \therefore\ b_2 \geq 2$.

(b) Assume $P_n > 0$, take $D = \sum n_i C_i \in |nK_X|$, $n_i > 0$

$K_X^2 < 0 \Longrightarrow K_X.D < 0 \quad \therefore\ K_X.C_i < 0$ for some i.

$$\overset{\|}{n_i C_i^2 + \sum_{j\neq i} n_j \underset{\overset{\vee\!\vert}{0}}{\underline{C_i.C_j}}} \Longrightarrow C_i^2 < 0$$

$$2g_{C_i} - 2 = \underset{\overset{\wedge}{0}}{C_i^2} + \underset{\overset{\wedge}{0}}{K_X.C_i}$$ adjunction formula

$\Longrightarrow C_i$ is an exceptional curve $\longrightarrow\!\!\!\!\!\!\times$

Therefore, $P_n \equiv 0$. If $q=0$, then X is rational
from the criterion of Castelnuovo.

Only minimal rational surfaces are $\mathbb{P}^2$ or $\mathbb{F}_{n\neq 1}$.

$\Longrightarrow K_X^2 = 9$ or 8 $\rightarrow\!\leftarrow$

Proposition 1: X minimal w/ $K_X^2 < 0$, then $\underbrace{X \text{ is ruled}}_{\exists \text{ a section}}$.

pf: X minimal w/ $K_X^2 < 0 \xRightarrow[\text{Lemma 1 (b)}]{} p_g = 0,\ q \geq 1$

$\Longrightarrow X \xrightarrow{\alpha} B = \alpha(X)$ Albanose map
w/ connected fibres
B smooth curve of $g_B = q$.

Assume that X is not ruled.

Claim 1: $C \subseteq X$ irreducible curve w/ $K_X . C < 0$, $|K + C| = \phi$
then $\alpha|_C$ is étale and is an isomorphism if $q \geq 2$.

In particular, $g_C = q$.

Claim 2: $\exists C \subseteq X$ irreducible w/ $K_X . C < -1$, $|K_X + C| = \phi$.

Assuming the claims, we want to lead to a contradiction.

- $q \geq 2 \xRightarrow[\text{Claim 1}]{} C$ is a section.

$$\underset{\substack{\| \\ 0}}{h^0(C)} + \underset{\substack{\| \ \because\ |K_X+C|=\phi \\ 0}}{h^0(K-C)} \geqslant \underset{\substack{\| \\ 1-q+\cancel{p_g}^{\,0}}}{\chi(\mathcal{O}_X)} + \tfrac{1}{2}C.(C-K_X)$$

$$= \boxed{\underset{\substack{/\!/ \\ 0}}{1-q} + \underbrace{\tfrac{1}{2}C.(C+K_X)}_{\substack{\| \\ g_C-1}}} - C.K_X \geqslant 2 \quad \text{adjunction}$$

Thus, C is movable.

F is the fibre of α, then $C \cap F \sim \underset{\substack{\uparrow \\ |C|}}{C'} \cap F \Rightarrow F$ rational

$\Rightarrow X$ ruled

- $q=1$, $\alpha|_C$ étale.

$\rightsquigarrow$

$$\begin{tikzcd}
 & X' \arrow[d, equal] & \\
C' \arrow[r, phantom, "\overset{i}{\subseteq}"] & X\times_B C \arrow[r, "\substack{\pi \\ \text{étale}}"] \arrow[d, "\alpha'"] & X \arrow[d, "\alpha"] \\
 & C \arrow[r] \arrow[ul, "s"', "\text{section}"] & B
\end{tikzcd}$$

$K_{X'} \cong \pi^* K_X$

$\chi_{top}(X') = n\,\chi_{top}(X)$

$\xrightarrow[\text{Noether's formula}]{} \chi(\mathcal{O}_{X'}) = n\,\chi(\mathcal{O}_X)$

$$h^0(X', C') + h^0(X', K_{X'}-C') \geqslant \underset{\substack{\| \\ n\chi(\mathcal{O}_X) \\ \| \ \because\ q=1 \\ 0 \quad p_g=0}}{\chi(\mathcal{O}_{X'})} + \underbrace{\tfrac{1}{2}C'.(C'-K_{X'})}_{\substack{\| \\ 2g_{C'}-2-K_{X'}.C' \geqslant 2}} \quad \text{adjunction}$$

- $K_{X'}.C' = \deg_{C'}(i^*K_{X'}) = \deg_C((\pi\circ s)^* K_X) = \deg_C(K_X) = \underline{K_X.C < -1}$
- $g_{C'} = 1 \quad \because\ C' \xrightarrow{\text{étale}} B,\ g_B = 1$
- $h^0(X', K_{X'}-C') = 0 \Leftarrow h^0(X', K_{X'}) = 0 \Longleftarrow K_{X'}.C' < 0$

Then C' is movable, following the same argument earlier. $X \longrightarrow C$ is ruled. Thus, $X \longrightarrow B$ is ruled. $\longrightarrow\!\!\!\!\!\!\times$

Proof of Claim 1:

$$0 = h^0(K_X+C) + \underset{\overset{\|}{0}}{h^0(-C)} \geqslant \underset{\overset{\|}{1-q+\cancel{p_g}^{\nearrow 0}}}{\chi(\mathcal{O}_X)} + \frac{1}{2}\underset{\overset{\|}{g_C-1}}{C.(C+K_X)} = g_C - q$$

$$\Rightarrow \quad g_C \leqslant q$$

X minimal $\Longrightarrow C^2 \geqslant 0$ $C.K_X < 0, \; C^2 < 0 \xRightarrow[\text{adjunction}]{} C.K_X = C^2 = -1$

$\Longrightarrow$ C can't be a proper subset of a fibre.
otherwise $C^2 < 0$

If C is a fibre, then $C^2 = 0$.
then $2g_C - 2 = C^2 + C.K_X \Rightarrow g_C = 0$ but X not ruled.

Therefore, $\alpha|_C :$

$$\begin{tikzcd} C \arrow[r, two heads] & B \\ N \arrow[u, "\text{normalization}"] \arrow[ur, "\text{degree } d"'] & \end{tikzcd}$$

Hurwitz formula $2-2g_N = d(2-2\underset{\overset{\|}{q}}{g_B}) - r$ ramification

$$\therefore \; q \geqslant g_C \geqslant g_N = 1 + d(q-1) + \frac{r}{2}$$

$$\Longrightarrow \left(q=1 \text{ or } d=1, \; \underbrace{g_C = g_N}_{\text{i.e } C \text{ smooth}}\right) \& \; r=0 .$$

Proof of Claim 2:

X minimal, $K^2 < 0 \xrightarrow[\text{in Lecture 7}]{\text{Lemma 1}} \exists D \geqslant 0$, s.t $K_X.D \leq -a$
$|K_X+D| = \phi$

take $\underset{\overset{\|}{2}}{a} > 0$

Write $D = \sum_i n_i C_i$, $n_i > 0$, $K_X . D < -1$

Removing the component C_i from D if $K_X . C_i \geq 0$,

and we claim that D is irreducible, of multiplicity 1.

- If $n_i \geq 2$, then $|K_X + 2C_i| = \phi$, $K_X . C_i < 0$

$$0 = h^0(K_X+2C_i) \geq \underbrace{\chi(\mathcal{O}_X)}_{\substack{\| \\ 1-q+\cancel{p_g}\nearrow 0}} + \cancel{\tfrac{1}{2}}\underbrace{(K_X+2C_i).\cancel{2}C_i}_{\substack{\| \\ \underbrace{2C_i.(C_i+K_X)}_{4(q-1)} - \underbrace{C_i.K_X}_{\geq 0}}} > 3(q-1) \geq 0 \qquad \leftarrow\!\!\!\!\!\times$$

- If $|K_X + C_i + C_j| = \phi$, $K_X . C_i < 0$, $K_X . C_j < 0$

$$\begin{aligned} 0 = h^0(K_X + C_i + C_j) &= 1-q + \tfrac{1}{2}(K_X + C_i + C_j).(C_i + C_j) + h^1(K_X + C_i + C_j) \\ &= 1-q + \tfrac{1}{2}\underbrace{C_i.(C_i+K_X)}_{g_{C_i}-1 = q-1 \text{ claim 1}} + \tfrac{1}{2}\underbrace{C_j.(C_j+K_X)}_{g_{C_j}-1 = q-1 \text{ claim 1}} + C_i.C_j \\ &\quad + h^1(K_X+C_i+C_j) \\ &= q-1 + h^1(K_X+C_i+C_j) + C_i.C_j \end{aligned}$$

$$\Longrightarrow \qquad h^1(K_X+C_i+C_j) = C_i.C_j = 0$$

$\because q \geq 1$ from Lemma 1 (b)

$$0 \longrightarrow \mathcal{O}_X(-C_i-C_j) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C_i} \oplus \mathcal{O}_{C_j} \longrightarrow 0$$

$$\underset{\substack{\wr\| \\ k}}{H^0(X,\mathcal{O}_X)} \longrightarrow \underset{\substack{\wr\| \\ k}}{H^0(C_i,\mathcal{O}_{C_i})} \oplus \underset{\substack{\wr\| \\ k}}{H^0(C_j,\mathcal{O}_{C_j})} \longrightarrow \underset{\substack{\wr\| \\ H^1(X,\, K_X-C_i-C_j)^*}}{H^1(X, -C_i-C_j)} = 0 \qquad \leftarrow\!\!\!\!\!\times$$

$\square$

We still need to understand the case when

$K^2 = 0$, $p_g = 0$, $q = 1$

Proposition 2: X minimal surface w/ $K^2=0$, $p_g=0$, $q=1$

$X \xrightarrow{\alpha} \alpha(X)=B$ Albanese map

$g=$ genus of a generic fibre

then either ① $g\geqslant 2$ & α is smooth, or

② $g=1$, singular fibre $F=nE$

E smooth elliptic curve.

pf: Lemma 1 (a) $\Longrightarrow$ $b_2=2$

- Fibres of α are irreducible.

  If C_i, C_j are distinct irreducible components of a fibre.

  Claim: $C_i, C_j, \overset{\text{very ample}}{H}$ are linear independent in $H^2(X,\mathbb{Z})$

  Otherwise $aC_i+bC_j+cH=0$.

  If $c\neq 0$, then $H.F\leq 0$ $\rightarrow\!\!\!\!\!\times$

  Therefore, $c=0$ or $C_i=rC_j$. $r\in\mathbb{Q}$.

  $0<H.C_i=rH.C_j \Longrightarrow r>0$ $\rightarrow\!\!\!\!\!\times$

  $0>C_i.C_i=r\underbrace{C_i.C_j}_{\geqslant 0} \Longrightarrow r<0$

- Assume that $F_s=nC$ is a multiple fibre

  F_η is a generic fibre.

$$\chi_{top}(F_s)=\underline{\chi_{top}(C)\geqslant 2\chi(\mathcal{O}_C)} \quad \text{Lemma 2.}$$

$$2\chi(\mathcal{O}_C)\underset{\text{adjunction}}{=}-C.(C+K_X)=-\frac{1}{n}F_\eta.K_X=\frac{2}{n}\chi(\mathcal{O}_{F_\eta})\underset{\text{Lemma 2}}{=}\frac{1}{n}\chi_{top}(F_\eta)$$

$$\Longrightarrow \quad \chi_{top}(F_s) \geqslant \frac{1}{n}\underset{\substack{\wedge\!\shortmid \\ 0}}{\chi_{top}(F_\eta)} \geqslant \chi_{top}(F_\eta)$$

$g \geqslant 1$

$$\therefore \quad \chi_{top}(F_s) = \chi_{top}(F_\eta) \quad \text{iff} \quad \chi_{top}(C) = 2\chi(\mathcal{O}_C)$$

C smooth

$$\frac{1}{n}\chi_{top}(F_\eta) = \chi_{top}(F_\eta)$$

$n=1$ or $g=1$

$$\underset{\substack{\shortparallel \\ 1-q+p_g \\ \shortparallel \\ 0}}{\chi(\mathcal{O}_X)} = \frac{1}{12}\Big(\underset{\substack{\shortparallel \\ 0}}{K_X^2} + \underset{\substack{\shortparallel \\ \cancel{\chi_{top}(B)}\,\nearrow 0\,\cdot\,\chi_{top}(F_\eta) + \sum_s\Big(\underset{\substack{\equiv \\ 0}}{\boxed{\chi_{top}(F_s) - \chi_{top}(F_\eta)}}\Big)}}{\chi_{top}(X)}\Big)$$

$\forall s$

$\square$

Lemma 2: Let C be a reduced, possibly reducible curve

then $\chi_{top}(C) \geqslant 2\chi(\mathcal{O}_C)$, $=$ iff C is smooth.

pf: $n: N \longrightarrow C$ normalization. $\Longrightarrow$ $H^i(N, \mathcal{F}) \cong H^i(C, n_*\mathcal{F})$,

n: affine

for quasi-coheren $\mathcal{F}$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}_C \arrow[r] \arrow[d, hook] & n_*\mathbb{C}_N \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\Rightarrow"] & \delta \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] & n_*\mathcal{O}_N \arrow[r] & \epsilon \arrow[r] & 0
\end{tikzcd}$$

$n_*\mathbb{C}_N \cap \mathcal{O}_C = \mathbb{C}_C$

$$\chi_{top}(C) = \underset{\chi_{top}(N) \,=\, 2\chi(\mathcal{O}_N)}{\underline{\chi(n_*\mathbb{C}_N)}} - \chi(\delta) \qquad \chi(\mathcal{O}_C) = \underset{\substack{\shortparallel \\ \chi(\mathcal{O}_N)}}{\underline{\chi(n_*\mathcal{O}_N)}} - \chi(\epsilon)$$

N: smooth
curve Riemann-Roch

$$\begin{aligned}
\therefore \quad \chi_{top}(C) &= 2\chi(\mathcal{O}_N) - \chi(\delta) = 2\big(\chi(\mathcal{O}_C) + \chi(\epsilon)\big) - \chi(\delta) \\
&= 2\chi(\mathcal{O}_C) + 2\chi(\epsilon) - \chi(\delta) \\
&\geqq 2\chi(\mathcal{O}_C) + \chi(\epsilon)
\end{aligned}$$

$\chi(\epsilon) \geqslant \chi(\delta)$

Remark: In the second case of the Proposition 2, one can do base change $z \to z^m$ for each multiple fibre, then

$$\begin{tikzcd} mE = X_p \arrow[d] \arrow[r, phantom, "\subseteq"] & X \arrow[d] & \tilde{X} \arrow[l] \arrow[d] & E \arrow[l, hook'] \arrow[d] \\ p \arrow[r, phantom, "\in"] & C & \tilde{C} \arrow[l] \arrow[r, phantom, "\ni"] & \tilde{p} \end{tikzcd}$$

reduced

$z^m \mapsfrom z$ locally.

One can actually do better.

Proposition 3: $\underset{\text{surface}}{X} \xrightarrow{p} \underset{\text{curve}}{B}$ smooth fibration s.t. ① $g_B = 1$, $g_F \geq 1$ or ② $g_F = 1$

then $\exists\, B' \xrightarrow[\text{étale}]{} B$ s.t.

$$\begin{tikzcd} X \arrow[d] & X' = X \times_B B' \arrow[l] \arrow[d] \\ B \arrow[ur, phantom, "\square"] & B' \arrow[l] \end{tikzcd}$$

trivial fibration

& $X \cong B' \times F / G$

G: finit group

pf: A family of J_n-rigidified curve of genus g over T is $\begin{tikzcd} X \arrow[d, "f"'] \\ T \end{tikzcd}$ w/ $R^1 f_* \mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$

Given a curve of genus g over T

$$\begin{tikzcd} X \arrow[d, "f"'] \\ T \ni t \end{tikzcd} \quad \text{w/} \quad \begin{tikzcd} \pi_1(T,t) \arrow[r] \arrow[dr] & \mathrm{Aut}(H^1(X, \mathbb{Z}/n\mathbb{Z})) \\ & \overset{\psi}{0} \end{tikzcd}$$

Otherwise, $|H^1(X_t, \mathbb{Z}/n\mathbb{Z})| < \infty$ thus $|\mathrm{Aut}(H^1(X_t, \mathbb{Z}/n\mathbb{Z}))| < \infty$

$$\begin{tikzcd} \pi_1(T,t) \arrow[r] & \mathrm{Aut}(H^1(X, \mathbb{Z}/n\mathbb{Z})) \\ N \arrow[u, phantom, "\triangledown"] \arrow[r] & 0 \arrow[u, phantom, "\psi"] \end{tikzcd}$$

finite index

$\Longrightarrow \exists\ T' \xrightarrow[\text{étale}]{} T$ s.t

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
T' \arrow[r] & T
\end{tikzcd}$$

is J_n-rigidified.

Fact : ① $n \geqslant 3$. J_n-rigidified curves have no automorphisms.

$\rightsquigarrow\ \exists$

$$\begin{tikzcd}
\mathcal{C}_{g,n} \arrow[d] \\
T_{g,n}
\end{tikzcd}$$

$\mathcal{C}_{g,n}$ universal curve

$T_{g,n}$ moduli space of J_n-rigidified curves

s.t

$$\begin{tikzcd}
X \cong T \times_{T_{g,n}} \mathcal{C}_{g,n} \arrow[r] \arrow[d] & \mathcal{C}_{g,n} \arrow[d] \\
T \arrow[r, "\exists!"] & T_{g,n}
\end{tikzcd}$$

② $g \geqslant 2$, no non-constant $h : \mathbb{C} \longrightarrow T_{g,n}$

universal cover of $T_{g,n}$ is a bounded domain.

$g = 1$, no non-constant $h : T \longrightarrow T_{1,n}$. T: compact

j-invariant constant $\Rightarrow$ trivial family

smooth fibration

Proposition 2 + Proposition 3 implies the following:

Corollary 1 : X = minimal, non-ruled, w/ $p_g = 0$. $q = 1$, $K^2 = 0$

$\Longrightarrow\ \exists$ • curves B, F . $g_B, g_F \geqslant 1$.

• finite group $G \leq \mathrm{Aut}(B)$

$$\begin{tikzcd}
G \times (B \times F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & (B \times F) \arrow[d] \\
G \times B \arrow[r] & B
\end{tikzcd}$$

s.t $X \cong (B\times F)/G$, $B/G \cong$ elliptic curve

If $g(F) \geq 2$, then B elliptic , ? G group of translation

More precisely, we have the following:

Theorem 1: $X =$ minimal non-ruled surface w/ $p_g = 0$, $q \geq 1$

then $X \cong (B\times F)/G$, B, F irrational curves

G finite group acting faithfully on B, F,

B/G elliptic , F/G rational s.t

- B elliptic , G group of translation of B, or
- F elliptic , G acts freely on $B\times F$.

Conversely, any surface w/ above properties is minimal

w/ $p_g = 0$, $q = 1$, $K^2 = 0$ & non-ruled.